PERIODIC WAVES GENERATED BY A SOURCE LOCATED ABOVE A SLOPING BOTTOM

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In recent years a large number of investigations devoted to consideration of the wave motions of a fluid in basins which have a plane bottom inclined to the horizon have been published. These investigations are described in sufficient detail in Stoker's book [1] and in Wehausen's long review article [2].

The problem of waves excited on the surface of a fluid by a periodically acting source located at a given depth above a uniformly inclined bottom is considered below. The investigation is carried out for plane-parallel potential motions, using methods presented in [1,2].

1. Let us assume that the angle α of the inclination of the bottom to the horizon is an integral fraction of $90°$, i.e. $\alpha = \pi/2n$.

We shall call the characteristic stream function $w(z, t)$

$$w(z, t) = \varphi(x, y; t) + i\psi(x, y; t)$$

If the axis Ox is drawn along the average water level, the axis Oy directed vertically upwards and the point O taken at the intersection of the free water level with the bottom of the basin, the following boundary condition will then obtain for all positive values of x:

$$\left(\frac{\partial^2\varphi}{\partial t^2} + g\frac{\partial\varphi}{\partial y}\right)_{y=0} = 0$$

Let us consider periodic motions with frequency σ; we shall assume

$$w(z, t) = f(z) \cos \sigma t, \qquad f(z) = \varphi(x, y) + i\psi(x, y)$$
$$\varphi(x, y; t) = \varphi(x, y) \cos \sigma t, \qquad \psi(x, y; t) = \psi(x, y) \cos \sigma t$$

The new function $\varphi(x, y)$ will satisfy the condition

$$\left(\frac{\partial \varphi}{\partial y} - \nu\varphi\right)_{y=0} = 0 \qquad \left(\nu = \frac{\sigma^2}{g}\right)$$

Let us rewrite this condition in the following form:

$$\operatorname{Im}(df/dz + i\nu f) = 0 \quad \text{for } z = x \tag{1.1}$$

In addition, the function $\varphi(x, y)$ must also satisfy the following boundary condition:

$$\frac{\partial \varphi}{\partial x} \sin \alpha + \frac{\partial \varphi}{\partial y} \cos \alpha = 0 \quad \text{for } y = -x \tan \alpha$$

This last condition can be rewritten as

$$\operatorname{Im}(e^{i\alpha}\, df/dz) = 0 \quad \text{for } \arg z = -\alpha \tag{1.2}$$

Besides these conditions, we shall require the function $f(z)$ to be holomorphic about the point $z = 0$.

Let us assume that the waves on the surface of the fluid arise from a source at $z = \rho e^{-\mu i} = c_0$ having a periodic output $Q = q \cos \sigma t$. By virtue of this assumption the function $f(z)$ will have the following form in the neighborhood of the point c_0:

$$f(z) = \frac{q}{2\pi} \ln(z - c_0) + \ldots \tag{1.3}$$

Thus, the analytic function $f(z)$ must satisfy conditions (1.1) and (1.2), must be holomorphic at the point $z = 0$ and must have a logarithmic singularity at the point c_0. To determine the function $f(z)$ in accordance with these conditions let us consider the following function of a complex variable:

$$G(z) = \sum_{k=0}^{n-1} a_k \left[f^{(k+1)}(z) + i\nu f^{(k)}(z)\right]$$

The coefficients a_k here have the following values:

$$a_0 = 1, \qquad a_k = \frac{1}{\nu^k} \cot \alpha \cot 2\alpha \ldots \cot k\alpha \qquad (k = 1, 2, \ldots, n-1)$$

With these coefficients the function $G(z)$ will have real values at $z = x$ and $z = re^{-2\alpha i}$ (r is an arbitrary real number greater than zero [2].

If the function $f(z)$ had no singular points within or on the boundary of the basin, the function $G(z)$ would then be equal to zero on the whole plane of the complex variable z. But in the case under consideration the function $f(z)$ has a singularity at the point $z = c_0$. Let us find the function $G(z)$ in this case also.

Near the point $z = c_0$ the function $f(z)$ has the form (1.3); hence, it follows that near this same point the function $G(z)$ will have the form

$$G(z) = \frac{ivq}{2\pi}\ln(z-c_0) + \frac{q}{2\pi}\sum_{k=0}^{n-1}\frac{b_k}{(z-c_0)^{k+1}} - \frac{ivq}{2\pi}\sum_{k=1}^{n-1}\frac{b_k}{(z-c_0)^k} + \ldots \qquad (1.4)$$

where

$$b_k = (-)^k k!\, a_k$$

Let us now construct a function $g(z)$ that would be holomorphic everywhere within the sector $-2\alpha \leqslant \arg z \leqslant 0$ with the exception of the points $z = c_0 = \rho e^{-\mu i}$ and $z = c'_{2n-1} = \rho e^{\mu i + 2\alpha i(2n-1)}$ and would take real values for $\arg z = 0$ and $\arg z = -2\alpha$; in addition, the desired function must have the form (1.3) near the point $z = c_0$ and the analogous form

$$g(z) = \frac{ivq}{2\pi}\ln(z-c'_{2n-1}) + \frac{q}{2\pi}\sum_{k=0}^{n-1}\frac{b_k}{(z-c'_{2n-1})^{k+1}} - \frac{ivq}{2\pi}\sum_{k=0}^{n-1}\frac{b_k}{k(z-c'_{2n-1})^k} + \ldots$$

near the point c'_{2n-1}.

It can be shown that such a function will be defined by the following formula:

$$\begin{aligned} g(z) = {} & \frac{ivq}{2\pi}\ln\frac{(z-\rho e^{-\mu i})(ze^{-2\alpha i}-\rho e^{\mu i})(ze^{-4\alpha i}-\rho e^{-\mu i})\ldots(ze^{-2\alpha i(2n-1)}-\rho e^{\mu i})}{(z-\rho e^{\mu i})(ze^{-2\alpha i}-\rho e^{-\mu i})(ze^{-4\alpha i}-\rho e^{\mu i})\ldots(ze^{-2\alpha i(2n-1)}-\rho e^{-\mu i})} + \\ & + \frac{q}{2\pi}\sum_{k=0}^{n-1} b_k \sum_{j=0}^{2n-1}(-)^j\left[\frac{1}{(ze^{-2\alpha ij}-\rho e^{-\mu i})^{k+1}} + \frac{1}{(ze^{-2\alpha ij}-\rho e^{\mu i})^{k+1}}\right] - \\ & - \frac{ivq}{2\pi}\sum_{k=1}^{n-1}\frac{b_k}{k}\sum_{j=0}^{2n-1}(-)^j\left[\frac{1}{(ze^{-2\alpha ij}-\rho e^{-\mu i})^k} - \frac{1}{(ze^{-2\alpha ij}-\rho e^{\mu i})^k}\right] \end{aligned} \qquad (1.5)$$

In fact, let us consider the logarithmic term of this formula. If z has a real value, the fractions

$$\frac{z-\rho e^{-\mu i}}{z-\rho e^{\mu i}}, \quad \frac{ze^{-2\alpha i}-\rho e^{\mu i}}{ze^{-2\alpha i(2n-1)}-\rho e^{-\mu i}}, \quad \frac{ze^{-4\alpha i}-\rho e^{-\mu i}}{ze^{-2\alpha i(2n-2)}-\rho e^{\mu i}}, \ldots \frac{ze^{-2\alpha i(2n-1)}-\rho e^{\mu i}}{ze^{-2\alpha i}-\rho e^{-\mu i}}$$

by virtue of the fact that α is an integral fraction of 90°, will then be complex numbers with moduli equal to unity. Hence, it follows that the first term on the right-hand side of formula (1.5) has real values for z real.

Let us now take the following two terms from the first sum of formula (1.5):

$$\frac{(-)^j}{(ze^{-2\alpha ij}-\rho e^{-\mu i})^{k+1}}, \qquad \frac{(-)^{j'}}{(ze^{-2\alpha ij'}-\rho e^{\mu i})^{k+1}} \qquad \begin{pmatrix} j'=2n-j & \text{for } j>0 \\ j'=0 & \text{for } j=0 \end{pmatrix}$$

The sum of these two terms is a real number; consequently, the first sum of formula (1.5) has real values for $z = x$. It is possible to establish similarly that the second sum of this same formula is a real number for $z = x$ too. Thus, the function $g(z)$ has real values for $z = x$.

Let us now set $z = re^{-2\alpha i}$ in formula (1.5) and consider again the logarithmic term of this formula. The fractions

$$\frac{re^{-2\alpha i}-\rho e^{-\mu i}}{re^{-2\alpha i-2\alpha i\,(2n-2)}-\rho e^{\mu i}}, \qquad \frac{re^{-2\alpha i}e^{-2\alpha i}-\rho e^{\mu i}}{re^{-2\alpha i-2\alpha i\,(2n-3)}-\rho e^{-\mu i}}, \ldots$$

$$\frac{re^{-2\alpha i}e^{-2\alpha i\,(2n-2)}-\rho e^{-\mu i}}{re^{-2\alpha i}-\rho e^{\mu i}}, \qquad \frac{re^{-2\alpha i}e^{-2\alpha i\,(2n-1)}-\rho e^{\mu i}}{re^{-2\alpha i}e^{-2\alpha i\,(2n-1)}-\rho e^{-\mu i}}$$

represent complex numbers with moduli equal to unity.

Hence, it follows that the first term of the right-hand side of formula (1.5) will have real values for $z = re^{-2\alpha i}$.

Let us consider next both sums of formula (1.5). From the previous consideration of the logarithmic term it follows at the same time, i.e. by similar comparisons of the individual terms, that each of the sums of formula (1.5) has real values for $z = re^{-2\alpha i}$.

The function $g(z)$, defined by formula (1.5), satisfies the following conditions:

$$\operatorname{Im} g(z) = 0 \quad \text{for } z = x, \; z = re^{-2\alpha i} \tag{1.6}$$

Let us note also that the function $g(z)$ takes pure imaginary values for $z = re^{-\alpha i}$.

In fact, the first term of formula (1.5) is a pure imaginary number since each of the products

$$(re^{-\alpha i}-\rho e^{-\mu i})\,(re^{-\alpha i-2\alpha i\,(2n-1)}-\rho e^{\mu i})$$
$$(re^{-\alpha i}e^{-2\alpha i}-\rho e^{\mu i})\,(re^{-\alpha i-2\alpha i\,(2n-2)}-\rho e^{-\mu i})$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(re^{-\alpha i} - \rho e^{\mu i})\ (re^{-\alpha i - 2\alpha i\,(2n-1)} - \rho e^{-\mu i})$$
$$(re^{-\alpha i} e^{-2\alpha i} - \rho e^{-\mu i})\ (re^{-\alpha i - 2\alpha i\,(2n-2)} - \rho e^{\mu i})$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

is a real number. The terms of the first sums

$$(-)^j \left[\frac{1}{(ze^{-2\alpha i j} - \rho e^{-\mu i})^{k+1}} + \frac{1}{(ze^{-2\alpha i j} - \rho e^{\mu i})^{k+1}} \right]$$
$$(-)^{2n-1-j} \left[\frac{1}{(ze^{-2\alpha i\,(2n-1-j)} - \rho e^{-\mu i})^{k+1}} + \frac{1}{(ze^{-2\alpha i\,(2n-1-j)} - \rho e^{\mu i})^{k+1}} \right]$$

give by addition pure imaginary numbers if $z = re^{-\alpha i}$.

The terms of the second sums

$$(-)^j \left[\frac{1}{(ze^{-2\alpha i j} - \rho e^{-\mu i})^{k}} - \frac{1}{(ze^{-2\alpha i j} - \rho e^{\mu i})^{k}} \right]$$
$$(-)^{2n-1-j} \left[\frac{1}{(ze^{-2\alpha i\,(2n-1-j)} - \rho e^{-\mu i})^{k}} - \frac{1}{(ze^{-2\alpha i\,(2n-1-j)} - \rho e^{\mu i})^{k}} \right]$$

give real numbers upon their addition.

Hence, it follows that the function $g(z)$ has pure imaginary values for $z = re^{-\alpha i}$.

Let us take the designations

$$c_0 = \rho e^{-\mu i}, \quad c_1 = \rho e^{-\mu i + 2\alpha i}, \ldots, c_{2n-1} = \rho e^{-\mu i + 2\alpha i\,(2n-1)}$$
$$c_0{}' = \rho e^{\mu i}, \quad c_1{}' = \rho e^{\mu i + 2\alpha i}, \ldots, c'_{2n-1} = \rho e^{\mu i + 2\alpha i\,(2n-1)} \tag{1.7}$$

and consider the function

$$\chi(z) = \frac{i\nu q}{2\pi} \ln \frac{(z - c_0)(z - c_1{}')(z - c_2)(z - c_3{}') \ldots (z - c'_{2n-1})}{(z - c_0{}')(z - c_1)(z - c_2{}')(z - c_3) \ldots (z - c_{2n-1})} \tag{1.8}$$

which enters into formula (1.5).

The function $\chi(z)$ as well as the function $G(z)$ has real values for $z = x$ and $z = re^{-2\alpha i}$.

Thus, the single-valued function

$$G(z) - \chi(z) = \sum_{k=0}^{n-1} a_k\,[f^{(k+1)}(z) + i\nu f^{(k)}(z)] - \chi(z) \tag{1.9}$$

takes real values for $z = x$ and $z = re^{-2\alpha i}$.

The function which is equal to the sum of the last two terms of formula (1.5)

$$h(z)=\frac{q}{2\pi}\sum_{k=0}^{n-1}b_k\sum_{j=0}^{2n-1}(-)^j\left[\frac{1}{(ze^{-2\alpha ij}-\rho e^{-\mu i})^{k+1}}+\frac{1}{(ze^{-2\alpha ij}-\rho e^{\mu i})^{k+1}}\right]-$$
$$-\frac{ivq}{2\pi}\sum_{k=1}^{n-1}\frac{b_k}{k}\sum_{j=0}^{2n-1}(-)^j\left[\frac{1}{(ze^{-2\alpha ij}-\rho e^{-\mu i})^{k}}-\frac{1}{(ze^{-2\alpha ij}-\rho e^{\mu i})^{k}}\right] \tag{1.10}$$

also takes real values on the sides of the angle $z = 0$ and $z = re^{-2\alpha i}$.

Hence, it follows that the function (1.9) can be continued onto the entire complex plane and its values coincide* with the values of the function $h(z)$; thus, to determine the unknown function $f(z)$ we obtain the differential equation

$$\sum_{k=0}^{n-1} a_k\,[f^{(k+1)}(z)+ivf^{(k)}(z)]=\chi(z)+h(z) \tag{1.11}$$

2. Let us turn to the integration of equation (1.11). The corresponding homogeneous equation has as its characteristic equation the following equation:

$$\sum_{k=0}^{n-1} a_k\,(\lambda^{k+1}+iv\lambda^k)=0 \tag{2.1}$$

The roots of this equation are

$$\lambda_0=-iv,\ \lambda_1=-iv\varkappa,\ \lambda_2=-iv\varkappa^2,\ \ldots,\ \lambda_{n-1}=-iv\varkappa^{n-1}$$
$$(\varkappa=e^{-2\alpha i}=e^{-\pi i/n})$$

The general integral of the homogeneous equation is written as

$$f(z)=C_0e^{\lambda_0 z}+C_1e^{\lambda_1 z}+C_2e^{\lambda_2 z}+\ldots+C_{n-1}e^{\lambda_{n-1}z} \tag{2.2}$$

The particular solution of the inhomogeneous equation (1.11) is found by the method of variation of parameters.

Letting $C_0, C_1, C_2, \ldots, C_{n-1}$ in (2.2) be unknown functions of the variable z, we obtain

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* In fact, these functions can differ from one another by a pure imaginary constant number which, however, can be taken equal to zero.

$$\frac{dC_0}{dz}e^{\lambda_0 z}\lambda_0 + \frac{dC_1}{dz}e^{\lambda_1 z}\lambda_1{}^k + \frac{dC_2}{dz}e^{\lambda_2 z}\lambda_2{}^k + \ldots + \frac{dC_{n-1}}{dz}e^{\lambda_{n-1} z}\lambda_{n-1}^k = 0$$
$$(k = 0, 1, 2, \ldots, n-2)$$
$$\frac{dC_0}{dz}e^{\lambda_0 z}\lambda_0^{n-1} + \frac{dC_1}{dz}e^{\lambda_1 z}\lambda_1^{n-1} + \frac{dC_2}{dz}e^{\lambda_2 z}\lambda_2^{n-1} + \ldots + \frac{dC_{n-1}}{dz}e^{\lambda_{n-1} z}\lambda_{n-1}^{n-1} = \nu^{n-1} g(z)$$

The solution of this system of equations has the form

$$\frac{dC_0}{dz} = \nu^{n-1} g(z)\frac{\Delta_0}{\Delta}e^{-\lambda_0 z}, \quad \frac{dC_1}{dz} = \nu^{n-1} g(z)\frac{\Delta_1}{\Delta}e^{-\lambda_1 z}, \quad \ldots \quad \frac{dC_{n-1}}{dz} = \nu^{n-1} g(z)\frac{\Delta_{n-1}}{\Delta}e^{-\lambda_{n-1} z}$$

$$\Delta = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \lambda_0 & \lambda_1 & \ldots & \lambda_{n-1} \\ \lambda_0{}^2 & \lambda_1{}^2 & \ldots & \lambda_{n-1}^2 \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_0^{n-1} & \lambda_1^{n-1} & \ldots & \lambda_{n-1}^{n-1} \end{vmatrix}$$

Here Δ_0, Δ_1, Δ_2, ..., Δ_{n-1} are the minors of the determinant Δ which correspond to the elements of the last line and which are taken with the corresponding signs.

We have

$$\Delta = \Delta_0(\lambda_0 - \lambda_1)(\lambda_0 - \lambda_2)\ldots(\lambda_0 - \lambda_{n-1})$$
$$\Delta = \Delta_1(\lambda_1 - \lambda_0)(\lambda_1 - \lambda_2)\ldots(\lambda_1 - \lambda_{n-1})$$
$$\Delta = \Delta_2(\lambda_2 - \lambda_0)(\lambda_2 - \lambda_1)\ldots(\lambda_2 - \lambda_{n-1})$$
$$\ldots\ldots\ldots$$
$$\Delta = \Delta_{n-1}(\lambda_{n-1} - \lambda_0)(\lambda_{n-1} - \lambda_1)\ldots(\lambda_{n-1} - \lambda_{n-2})$$

Let us take the notation

$$\Lambda(\lambda) = (\lambda - \lambda_0)(\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_{n-1})$$

With this notation we have

$$\frac{\Delta}{\Delta_0} = \Lambda'(\lambda_0), \quad \frac{\Delta}{\Delta_1} = \Lambda'(\lambda_1), \ldots, \quad \frac{\Delta}{\Delta_{n-1}} = \Lambda'(\lambda_{n-1})$$

Consequently

$$\frac{dC_0}{dz} = \frac{\nu^{n-1}}{\Lambda'(\lambda_0)} g(z)e^{-\lambda_0 z}, \qquad \frac{dC_1}{dz} = \frac{\nu^{n-1}}{\Lambda'(\lambda_1)} g(z)e^{-\lambda_1 z}, \ldots,$$
$$\frac{dC_{n-1}}{dz} = \frac{\nu^{n-1}}{\Lambda'(\lambda_{n-1})} g(z)e^{-\lambda_{n-1} z}$$

Hence, denoting the arbitrary constants by K_0, K_1, ..., K_{n-1}, we have

$$C_0(z) = K_0 + \frac{\nu^{n-1}}{\Lambda'(\lambda_0)} \int_0^z g(\zeta)\, e^{-\lambda_0\zeta}\, d\zeta, \qquad C_1(z) = K_1 + \frac{\nu^{n-1}}{\Lambda'(\lambda_1)} \int_0^z g(\zeta)\, e^{-\lambda_1\zeta}\, d\zeta \tag{2.3}$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$C_{n-1}(z) = K_{n-1} + \frac{\nu^{n-1}}{\Lambda'(\lambda_{n-1})} \int_0^z g(\zeta)\, e^{-\lambda_{n-1}\zeta}\, d\zeta$$

The resultant expression for the function $f(z)$ will have the following form:

$$f(z) = K_0 e^{\lambda_0 z} + K_1 e^{\lambda_1 z} + \ldots + K_{n-1} e^{\lambda_{n-1} z} + \frac{\nu^{n-1}}{\Lambda'(\lambda_0)} e^{\lambda_0 z} \int_0^z g(\zeta)\, e^{-\lambda_0\zeta}\, d\zeta +$$

$$+ \frac{\nu^{n-1}}{\Lambda'(\lambda_1)} e^{\lambda_1 z} \int_0^z g(\zeta)\, e^{-\lambda_1\zeta}\, d\zeta + \ldots + \frac{\nu^{n-1}}{\Lambda'(\lambda_{n-1})} e^{\lambda_{n-1} z} \int_0^z g(\zeta)\, e^{-\lambda_{n-1}\zeta}\, d\zeta \tag{2.4}$$

Let us note the formulas for the derivatives $\Lambda'(\lambda_0)$, $\Lambda'(\lambda_1)$, ..., $\Lambda'(\lambda_{n-1})$

$$\begin{aligned}\Lambda'(\lambda_0) &= (\lambda_0 - \lambda_1)(\lambda_0 - \lambda_2)\ldots(\lambda_0 - \lambda_{n-1}) = \\ &= (-i\nu)^{n-1}(1-\varkappa)(1-\varkappa^2)\ldots(1-\varkappa^{n-1}) = \\ &= (2\nu)^{n-1} e^{-\frac{1}{4}\pi i(n-1)} \sin\alpha \sin 2\alpha \ldots \sin(n-1)\alpha \end{aligned} \tag{2.5}$$

$$\Lambda'(\lambda_s) = \left(\frac{i}{\varkappa}\right)^s \Lambda'(\lambda_0) \tan\alpha \tan 2\alpha \ldots \tan s\alpha \qquad (s = 1, 2, \ldots, n-1)$$

Let us also note the formulas for the conjugate quantities

$$\overline{\Lambda'(\lambda_0)} = e^{\frac{1}{2}\pi i(n-1)} \Lambda'(\lambda_0), \qquad \overline{\Lambda'(\lambda_s)} = -\frac{1}{\varkappa} \Lambda'(\lambda_{n-s-1}) \tag{2.6}$$

3. Let us determine the constants K_0, K_1, ..., K_{n-1} which enter into (2.4) so that condition (1.1) and the condition equivalent to (1.2)

$$\operatorname{Im} f(z) = 0 \quad \text{for } z = re^{-\alpha i} = z' \tag{3.1}$$

are satisfied.

Condition (3.1) can be rewritten as

$$f(z') - \overline{f(z')} = 0$$

From formula (2.4) we have

$$\overline{f(z')} = \overline{K_0 e^{\lambda_0 z'}} + \overline{K_1 e^{\lambda_1 z'}} + \ldots + \overline{K_{n-1} e^{\lambda_{n-1} z'}} +$$

$$+ \nu^{n-1} \left(\frac{e^{\lambda_0 z'}}{\Lambda'(\lambda_0)} \int_0^{z'} g(\zeta) e^{-\lambda_0 \zeta} d\zeta + \frac{e^{\lambda_1 z'}}{\Lambda'(\lambda_1)} \int_0^{z'} g(\zeta) e^{-\lambda_1 \zeta} d\zeta + \ldots + \frac{e^{\lambda_{n-1} z'}}{\Lambda'(\lambda_{n-1})} \int_0^{z} g(\zeta) e^{-\lambda_{n-1} \zeta} d\zeta \right)$$

The asterisk indicates that the expression within the parentheses should be replaced by its conjugate. Let us transform this expression using formulas (2.5), (2.6) and the following:

$$\overline{\lambda_0 z'} = \lambda_{n-1} z', \quad \overline{\lambda_1 z'} = \lambda_{n-2} z', \ldots, \overline{\lambda_{n-1} z'} = \lambda_0 z'$$

We obtain

$$\overline{f(z')} = \overline{K}_{n-1} e^{\lambda_0 z'} + \overline{K}_{n-2} e^{\lambda_1 z'} + \ldots + \overline{K}_0 e^{\lambda_{n-1} z'} -$$

$$- \nu^{n-1} \left(\frac{e^{\lambda_0 z'}}{\Lambda'(\lambda_0)} \int_0^{z'} \overline{g(\zeta)} e^{-\lambda_0 \zeta} d\zeta + \frac{e^{\lambda_1 z'}}{\Lambda'(\lambda_1)} \int_0^{z'} \overline{g(\zeta)} e^{-\lambda_1 \zeta} d\zeta + \ldots + \frac{e^{\lambda_{n-1} z'}}{\Lambda'(\lambda_{n-1})} \int_0^{z'} \overline{g(\zeta)} e^{-\lambda_{n-1} \zeta} d\zeta \right)$$

Thus, we obtain

$$f(z') - \overline{f(z')} = (K_0 - \overline{K}_{n-1}) e^{\lambda_0 z'} + (K_1 - \overline{K}_{n-2}) e^{\lambda_1 z'} + \ldots + (K_{n-1} - \overline{K}_0) e^{\lambda_{n-1} z'} +$$

$$+ \nu^{n-1} \left(\frac{e^{\lambda_0 z'}}{\Lambda'(\lambda_0)} \int_0^{z'} \left[g(\zeta) + \overline{g(\zeta)} \right] e^{-\lambda_0 \zeta} d\zeta + \frac{e^{\lambda_1 z'}}{\Lambda'(\lambda_1)} \int_0^{z'} \left[g(\zeta) + \overline{g(\zeta)} \right] e^{-\lambda_1 \zeta} d\zeta + \right.$$

$$\left. + \ldots + \frac{e^{\lambda_{n-1} z'}}{\Lambda'(\lambda_{n-1})} \int_0^{z'} \left[g(\zeta) + \overline{g(\zeta)} \right] e^{-\lambda_{n-1} \zeta} d\zeta \right)$$

But, since the function $g(\zeta)$ has pure imaginary values for $\zeta = re^{-\alpha i}$, then

$$f(z') - \overline{f(z')} = (K_0 - \overline{K}_{n-1}) e^{\lambda_0 z'} + (K_1 - \overline{K}_{n-2}) e^{\lambda_1 z'} + \ldots + (K_{n-1} - \overline{K}_0) e^{\lambda_{n-1} z'}$$

Hence, it follows that in order to satisfy condition (3.1) all of the terms in parentheses must be equal to zero

$$K_0 - \overline{K}_{n-1} = 0, \quad K_1 - \overline{K}_{n-2} = 0, \ldots, \quad K_{n-1} - \overline{K}_0 = 0 \tag{3.2}$$

Let us now turn to condition (1.1). Using formula (2.4), we find

$$\frac{df}{dz} + i\nu f = K_0 (\lambda_0 + i\nu) e^{\lambda_0 z} + K_1 (\lambda_1 + i\nu) e^{\lambda_1 z} + \ldots + K_{n-1} (\lambda_{n-1} + i\nu) e^{\lambda_{n-1} z} +$$

$$+ \nu^{n-1} g(z) \sum_{p=0}^{n-1} \frac{1}{\Lambda'(\lambda_p)} + \nu^{n-1} \sum_{p=0}^{n-1} \frac{\lambda_p + i\nu}{\Lambda'(\lambda_p)} e^{\lambda_p z} \int_0^z g(\zeta) e^{-\lambda_p \zeta} d\zeta$$

The next to last sum is equal to zero. In addition $\lambda_0 + i\nu = 0$. Consequently

$$\frac{df}{dz} + i\nu f = K_1(\lambda_1 + i\nu) e^{\lambda_1 z} + K_2(\lambda_2 + i\nu) e^{\lambda_2 z} + \ldots +$$
$$+ K_{n-1}(\lambda_{n-1} + i\nu) e^{\lambda_{n-1} z} + \nu^{n-1} \sum_{p=1}^{n-1} \frac{\lambda_p + i\nu}{\Lambda'(\lambda_p)} e^{\lambda_p z} \int_0^z g(\zeta) e^{-\lambda_p \zeta} d\zeta$$

Let us demonstrate condition (1.1) using this formula. Taking into account that the function $g(\zeta)$ has real values for $z = x$, we obtain

$$\sum_{p=1}^{n-1} \left[K_p + \frac{\nu^{n-1}}{\Lambda'(\lambda_p)} \int_0^x g(\zeta) e^{-\lambda_p \zeta} d\zeta \right] (\lambda_p + i\nu) e^{\lambda_p x} -$$
$$- \sum_{p=1}^{n-1} \left[\overline{K}_p + \frac{\nu^{n-1}}{\overline{\Lambda'(\lambda_p)}} \int_0^x g(\zeta) e^{-\overline{\lambda}_p \lambda} d\zeta \right] (\overline{\lambda}_p - i\nu) e^{\overline{\lambda}_p x} = 0$$

Let us transform the left-hand side of this equation using the properties of the quantities λ_p and $\Lambda'(\lambda_p)$ as well as formulas (3.2). We obtain

$$\sum_{p=1}^{n-1} [K_p(\lambda_p + i\nu) - K_{p-1}(\lambda_p - i\nu)] e^{\lambda_p x} +$$
$$+ \nu^{n-1} \sum_{p=1}^{n-1} \left[\frac{\lambda_p + i\nu}{\Lambda'(\lambda_p)} + \varkappa \frac{\lambda_p - i\nu}{\Lambda'(\lambda_{p-1})} \right] e^{\lambda_p x} \int_0^x g(\zeta) e^{-\lambda_p \zeta} d\zeta = 0 \qquad (3.3)$$

Let us show that the quantities

$$A_p = \frac{\lambda_p + i\nu}{\Lambda'(\lambda_p)} + \varkappa \frac{\lambda_p - i\nu}{\Lambda'(\lambda_{p-1})}$$

are equal to zero. Substituting the expressions for λ_p here, we obtain a new representation for the quantities A_p

$$A_p = i\nu \left[\frac{1 - \varkappa^p}{\Lambda'(\lambda_p)} - \frac{\varkappa(1 + \varkappa^p)}{\Lambda'(\lambda_{p-1})} \right]$$

Substituting here for $\Lambda'(\lambda_p)$ and $\Lambda'(\lambda_{p-1})$ from expression (2.5), we find

$$A_p = \frac{\nu \varkappa^p}{i^{p-1} \Lambda'(\lambda_0)} [\cot \alpha (1 - \varkappa^p) - i(1 + \varkappa^p)] \cot \alpha \cot 2\alpha \ldots \cot (p-1)\alpha$$

But the quantity in the square brackets is equal to zero. Consequently, $A_p = 0$.

Formula (3.3) can now be written as

$$\sum_{p=1}^{n-1} [K_p(\lambda_p + i\nu) - K_{p-1}(\lambda_p - i\nu)]\, e^{\lambda_p x} = 0$$

Hence, it follows that

$$K_p(\lambda_p + i\nu) - K_{p-1}(\lambda_p - i\nu) = 0 \qquad (p = 1, 2, \ldots, n-1)$$

Substituting here the expression for λ_p, we find

$$K_p = iK_{p-1} \cot p\alpha$$

Consequently

$$K_1 = iK_0 \cot\alpha, \qquad K_2 = i^2 K_0 \cot\alpha \cot 2\alpha, \ldots$$

$$K_p = i^p K_0 \cot\alpha \cot 2\alpha \ldots \cot p\alpha, \ldots$$

$$K_{n-1} = i^{n-1} K_0 \cot\alpha \cot 2\alpha \ldots \cot(n-1)\alpha$$

Substituting these expressions for K into condition (3.2), we come to only a single consequence

$$K_0 = ae^{-\frac{1}{4}\pi i(n-1)}$$

where a is an arbitrary real number.

Having determined the constants K_0, K_1, ..., K_{n-1}, we obtain the resultant expression for the function

$$f(z) = \sum_{p=0}^{n-1} i^p \left(ae^{-\frac{1}{4}\pi i(n-1)} + \frac{(-)^p \varkappa^p \nu^{n-1}}{\Lambda'(\lambda_0)} \int_0^z g(\zeta)\, e^{-\lambda_p \zeta}\, d\zeta\right) e^{\lambda_p z} \cot\alpha \cot 2\alpha \ldots \cot p\alpha \tag{3.4}$$

Here $\Lambda'(\lambda_0)$ is given by formula

$$\Lambda'(\lambda_0) = (2\nu)^{n-1} e^{-\frac{1}{4}\pi i(n-1)} \sin\alpha \sin 2\alpha \ldots \sin(n-1)\alpha$$

for $n = 1$ it is necessary to take $\Lambda'(\lambda_0) = 1$.

4. Let us transform formula (3.4) into another form. The function $g(\zeta)$ is the sum of two terms $\chi(z)$ and $h(z)$ which are determined by formulas (1.8) and (1.10). Let us take the integral which enters into formula (3.4) and transform that part of it which corresponds to the function $\chi(z)$, i.e. let us take the integral

$$J = \int_0^z \chi(\zeta)\, e^{-\lambda_p \zeta}\, d\zeta$$

Using the formula for integration by parts, we obtain

$$J = -\frac{\chi(z)}{\lambda_p} e^{-\lambda_p z} - \frac{i\nu q}{2\pi\lambda_p} \sum_{j=0}^{2n-1} (-)^j \int_0^z \left(\frac{1}{\zeta - c_j} - \frac{1}{\zeta - c_j'}\right) e^{-\lambda_p \zeta} d\zeta$$

Formula (3.4) can now be represented in the following form:

$$f(z) = \sum_{p=0}^{n-1} i^p \left(ae^{-1/4\pi i(n-1)} + \frac{(-)^p \chi^p \nu^{n-1}}{\Lambda'(\lambda_0)} \int_0^z h(\zeta) e^{-\lambda_p \zeta} d\zeta\right) e^{\lambda_p z} \cot \alpha \ldots \cot p\alpha +$$
$$+ \frac{i\nu q \nu^{n-2}}{2\pi \Lambda'(\lambda_0)} \sum_{p=0}^{n-1} (-i)^{p-1} e^{\lambda_p z} \cot \alpha \cot 2\alpha \ldots \cot p\alpha \int_0^z e^{-\lambda_p \zeta} d\zeta \times$$
$$\times \sum_{j=0}^{2n-1} (-)^j \left(\frac{1}{\zeta - c_j} - \frac{1}{\zeta - c_j'}\right) - \chi(z) \frac{\nu^{n-2}}{\Lambda'(\lambda_0)} \sum_{p=0}^{n-1} (-i)^{p-1} \cot \alpha \cot 2\alpha \ldots \cot p\alpha \qquad (4.1)$$

It should be noted that the product $\cot \alpha \cot 2\alpha \ldots \cot p\alpha$ must be replaced by unity for $p = 0$.

With the help of formula (4.1) we find the equation of the wave surface

$$\eta = -\frac{\sigma}{g} \sin \sigma t \operatorname{Re} f(z), \qquad z = x \qquad (4.2)$$

Special interest lies in the form of the surface at distances far from the origin of the coordinate system, i.e. from the point of intersection of the average level of the fluid with the bottom of the basin. Therefore, let us give the variable $z = x$ in formula (4.1) an infinitely large positive value. For $x = \infty$ we have

$$\lim_{x=\infty} e^{\lambda_p x} \int_0^x h(\zeta) e^{-\lambda_p \zeta} d\zeta = 0 \qquad (p = 1, 2, \ldots, n-1)$$
$$\lim_{x=\infty} e^{\lambda_p x} \int_0^x e^{-\lambda_p \zeta} \sum_{j=0}^{2n-1} (-)^j \left(\frac{1}{\zeta - c_j} - \frac{1}{\zeta - c_j'}\right) d\zeta = 0$$
$$\lim_{x=\infty} \chi(x) = 0$$

Consequently, for large values of $z = x$ we have

$$f(x) = \left(ae^{-1/4\pi i(n-1)} + \frac{\nu^{n-1}}{\Lambda'(\lambda_0)} \int_0^\infty h(\zeta) e^{-\lambda_0 \zeta} d\zeta\right) e^{\lambda_0 x} -$$

$$-\frac{q\nu^{n-1}}{2\pi\Lambda'(\lambda_0)}e^{\lambda_0 x}\int_0^\infty e^{-\lambda_0\zeta}\,d\zeta\sum_{j=0}^{2n-1}(-)^j\left(\frac{1}{\zeta-c_j}-\frac{1}{\zeta-c_j'}\right)$$

Let us introduce the notation

$$A+Bi=\frac{\nu^{n-2}}{\Lambda'(\lambda_0)}\int_0^\infty\left[\nu h(\zeta)-\frac{q\nu}{2\pi}\sum_{j=0}^{2n-1}(-)^j\left(\frac{1}{\zeta-c_j}-\frac{1}{\zeta-c_j'}\right)\right]e^{-\lambda_0\zeta}\,d\zeta \quad (4.3)$$

We then obtain

$$f(x)=\{[A+a\cos{}^1/_4\,\pi(n-1)]+i\,[B-a\sin{}^1/_4\,\pi(n-1)]\}\times$$
$$\times(\cos\nu x-i\sin\nu x)$$

Hence, from formula (4.2) we find

$$\eta=-(\sigma/2g)\,[A+a\cos{}^1/_4\,\pi(n-1)]\,[\sin(\nu x+\sigma t)-\sin(\nu x-\sigma t)]+$$
$$+(\sigma/2g)\,[B-a\sin{}^1/_4\pi(n-1)]\,[\cos(\nu x+\sigma t)-\cos(\nu x-\sigma t)] \quad (4.4)$$

5. Through integration by parts formula (4.3) can be reduced to the form

$$A+Bi=\frac{q}{2\pi}\frac{\nu^{n-1}}{\Lambda'(\lambda_0)}\sum_{j=0}^{2n-1}\left(\frac{-1}{\varkappa}\right)^j F_j\,[(1-\varkappa^j)\,S(c_j)+(1+\varkappa^j)\,S(c_j')] \quad (5.1)$$

having introduced the designations

$$F_j=\sum_{k=0}^{n-1}\left(\frac{-i\nu}{\varkappa^j}\right)^k a_k,\qquad S_n(c)=\int_0^\infty\frac{e^{i\nu\zeta}\,d\zeta}{(\zeta-c)^n},\qquad S(c)=\int_0^\infty\frac{e^{i\nu\zeta}\,d\zeta}{\zeta-c}$$

Repeatedly using integration by parts, we find the formula

$$S_n(c)=\frac{1}{(n-1)!}\left\{(-)^{n-1}(n-2)!\,\frac{1}{c^{n-1}}+(-)^{n-2}(n-3)!\,\frac{i\nu}{c^{n-2}}+\right.$$
$$\left.+(-)^{n-3}(n-4)!\,\frac{(i\nu)^2}{c^{n-3}}+\ldots+(-)^1\frac{(i\nu)^{n-2}}{c}+(i\nu)^{n-1}S(c)\right\} \quad (5.2)$$

The integrand function of formula (4.3) can be represented as

$$h(\zeta)-\frac{q}{2\pi}\sum_{j=0}^{2n-1}(-)^j\left(\frac{1}{\zeta-c_j}-\frac{1}{\zeta-c_j'}\right)= \quad (5.3)$$
$$=\frac{q}{2\pi}\sum_{k=0}^{n-1}b_k\sum_{j=0}^{2n-1}(-)^j\varkappa^{-j(k+1)}\left[\frac{1}{(\zeta-c_j)^{k+1}}+\frac{1}{(\zeta-c_j')^{k+1}}\right]-$$

$$-\frac{i\nu q}{2\pi}\sum_{k=1}^{n-1}\frac{b_k}{k}\sum_{j=0}^{2n-1}(-)^j\varkappa^{-jk}\left[\frac{1}{(\zeta-c_j)^k}-\frac{1}{(\zeta-c_j')^k}\right]-\frac{q}{2\pi}\sum_{j=0}^{2n-1}(-)^j\left[\frac{1}{\zeta-c_j}-\frac{1}{\zeta-c_j'}\right]$$

Having noted this result, we obtain after applying the general formula

$$\int_0^\infty\sum_{j=0}^{2n-1}(-)^j\varkappa^{-j(k+1)}\left[\frac{1}{(\zeta-c_j)^{k+1}}+\frac{1}{(\zeta-c_j')^{k+1}}\right]e^{i\nu\zeta}d\zeta=$$

$$=\frac{(-)^k(k-1)!}{k!}\sum_{j=0}^{2n-1}(-)^j\varkappa^{-j(k+1)}\left(\frac{1}{c_j^k}+\frac{1}{c_j'^k}\right)+\frac{(-)^{k-1}(k-2)!}{k!}(i\nu)\times$$

$$\times\sum_{j=0}^{2n-1}(-)^j\varkappa^{-j(k+1)}\left(\frac{1}{c_j^{k-1}}+\frac{1}{c_j'^{k-1}}\right)+\frac{(-)^{k-2}(k-3)!}{k!}(i\nu)^2\times$$

$$\times\sum_{j=0}^{2n-1}(-)^j\varkappa^{-j(k+1)}\left(\frac{1}{c_j^{k-2}}+\frac{1}{c_j'^{k-2}}\right)+\ldots+\frac{(-)^1}{k!}(i\nu)^{k-1}\sum_{j=0}^{2n-1}(-)^j\varkappa^{-j(k-1)}\left(\frac{1}{c_j}+\frac{1}{c_j'}\right)+$$

$$+\frac{(i\nu)^k}{k!}\sum_{j=0}^{2n-1}(-)^j\varkappa^{-j(k+1)}\left[S(c_j)+S(c_j')\right]$$

By virtue of the value of κ all the sums which enter into the right-hand side of this formula with the exception of the last one vanish and thus

$$\int_0^\infty\sum_{j=0}^{2n-1}(-)^j\varkappa^{-j(k+1)}\left[\frac{1}{(\zeta-c_j)^{k+1}}+\frac{1}{(\zeta-c_j')^{k+1}}\right]e^{i\nu\zeta}d\zeta=$$

$$=\frac{(i\nu)^k}{k!}\sum_{j=0}^{2n-1}(-)^j\varkappa^{-j(k+1)}\left[S(c_j)+S(c_j')\right] \tag{5.4}$$

The formula

$$\int_0^\infty\sum_{j=0}^{2n-1}(-)^j\varkappa^{-jk}\left[\frac{1}{(\zeta-c_j)^k}-\frac{1}{(\zeta-c_j')^k}\right]e^{i\nu\zeta}d\zeta=\frac{(i\nu)^{k-1}}{(k-1)!}\sum_{j=0}^{2n-1}(-)^j\varkappa^{-jk}\left[S(c_j)-S(c_j')\right] \tag{5.5}$$

can be established as well.

At the same time we have

$$\int_0^\infty\sum_{j=0}^{2n-1}(-)^j\left(\frac{1}{\zeta-c_j}-\frac{1}{\zeta-c_j'}\right)e^{i\nu\zeta}d\zeta=\sum_{j=0}^{2n-1}(-)^j\left[S(c_j)-S(c_j')\right] \tag{5.6}$$

Using formulas (5.3), (5.4), (5.5) and (5.6), the equality

$$\int_0^\infty \left[h(\zeta) - \frac{q}{2\pi}\sum_{j=0}^{2n-1}(-)^j\left(\frac{1}{\zeta - c_j} - \frac{1}{\zeta - c_j'}\right)\right]e^{i\nu\zeta}\,d\zeta =$$
$$= \frac{q}{2\pi}\sum_{k=0}^{n-1}\frac{(i\nu)^k}{k!}\,b_k\sum_{j=0}^{2n-1}(-)^j\varkappa^{-j(k+1)}\left[S(c_j) + S(c_j')\right] -$$
$$- \frac{i\nu q}{2\pi}\sum_{k=1}^{n-1}\frac{(i\nu)^{k-1}}{(k-1)!}\frac{b_k}{k}\sum_{j=0}^{2n-1}(-)^j\varkappa^{-jk}\left[S(c_j) - S(c_j')\right] - \frac{q}{2\pi}\sum_{j=0}^{2n-1}(-)^j\left[S(c_j) - S(c_j')\right]$$

can be written.

Let us transpose the order of the summations on the right-hand side. We obtain

$$\int_0^\infty \left[h(\zeta) - \frac{q}{2\pi}\sum_{j=0}^{2n-1}(-)^j\left(\frac{1}{\zeta - c_j} - \frac{1}{\zeta - c_j'}\right)\right]e^{i\nu\zeta}\,d\zeta =$$
$$= \frac{q}{2\pi}\sum_{j=0}^{2n-1}\left(\frac{-1}{\varkappa}\right)^j(1-\varkappa^j)\,S(c_j)\sum_{k=0}^{n-1}\left(\frac{-i\nu}{\varkappa^j}\right)^k a_k +$$
$$+ \frac{q}{2\pi}\sum_{j=0}^{2n-1}\left(\frac{-1}{\varkappa}\right)^j(1+\varkappa^j)\,S(c_j')\sum_{k=0}^{n-1}\left(\frac{-i\nu}{\varkappa^j}\right)^k a_k$$

But the interior sums are equal to F_j. Hence we obtain formula (5.1).

6. Let us return to the results of Section 3. To the function $f(z)$ let us add a function which defines standing waves that depend on sin σt. For these waves we have

$$w_1(z, t) = f_1(z)\sin\sigma t$$

where

$$f_1(z) = be^{-\frac{1}{4}\pi i(n-1)}\left[e^{\lambda_0 z} + i\cot\alpha\, e^{\lambda_1 z} + i^2\cot\alpha\cot 2\alpha\, e^{\lambda_2 z} + \right.$$
$$\left. + \ldots + i^{n-1}\cot\alpha\cot 2\alpha\ldots\cot(n-1)\alpha\, e^{\lambda_{n-1}z}\right]$$

The real number b has an arbitrary value. In accordance with this function the elevation of the fluid at large values of x is determined by the formula

$$\eta_1 = (b\sigma/2g)\left[\cos\tfrac{1}{4}\pi(n-1)\cos(\nu x + \sigma t) - \sin\tfrac{1}{4}\pi\,(n-1)\sin(\nu x + \sigma t)\right] +$$
$$+ (b\sigma/2g)\left[\cos\tfrac{1}{4}\pi(n-1)\cos(\nu x - \sigma t) - \sin\tfrac{1}{4}\pi(n-1)\sin(\nu x - \sigma t)\right]$$

Let us add the ordinates η_1 to the ordinates of the wave surface represented by formula (4.4). We obtain

$$H = \left[\frac{b\sigma}{2g}\cos\frac{1}{4}\pi(n-1) + \frac{a\sigma}{2g}\sin\frac{1}{4}\pi(n-1) - \frac{\sigma B}{2g}\right]\cos(\nu x - \sigma t) +$$
$$+\left[\frac{a\sigma}{2g}\cos\frac{1}{4}\pi(n-1) - \frac{b\sigma}{2g}\sin\frac{1}{4}\pi(n-1) + \frac{\sigma A}{2g}\right]\sin(\nu x - \sigma t) +$$
$$+\left[\frac{b\sigma}{2g}\cos\frac{1}{4}\pi(n-1) - \frac{a\sigma}{2g}\sin\frac{1}{4}\pi(n-1) + \frac{\sigma B}{2g}\right]\cos(\nu x + \sigma t) -$$
$$-\left[\frac{a\sigma}{2g}\cos\frac{1}{4}\pi(n-1) + \frac{b\sigma}{2g}\sin\frac{1}{4}\pi(n-1) + \frac{\sigma A}{2g}\right]\sin(\nu x + \sigma t)$$

Let us set the condition that the waves must vanish at infinity. By virtue of this condition the coefficients of the last two trigonometric functions must vanish. Hence, we obtain the values of the arbitrary coefficients a and b

$$a = -A\cos\frac{\pi(n-1)}{4} + B\sin\frac{\pi(n-1)}{4},\ b = -A\sin\frac{\pi(n-1)}{4} - B\cos\frac{\pi(n-1)}{4}$$

For these values of a and b the equation of the wave is written as

$$H = -(\sigma B / g)\cos(\nu x - \sigma t) \quad \text{for } n \equiv 1 \pmod 4 \tag{6.1}$$

$$H = -(\sigma / g\sqrt{2})(A + B)\cos(\nu x - \sigma t + {}^1/_4\pi) \quad \text{for } n \equiv 2 \pmod 4 \tag{6.2}$$

$$H = (\sigma A / g)\sin(\nu x - \sigma t) \quad \text{for } n \equiv 3 \pmod 4 \tag{6.3}$$

$$H = (\sigma / g\sqrt{2})(A - B)\sin(\nu x - \sigma t + {}^1/_4\pi) \quad \text{for } n \equiv 0 \pmod 4 \tag{6.4}$$

The quantities A and B are given by formula (5.1).

7. Let us use the formulas obtained to investigate a series of special cases. Let us consider first waves which proceed from a source located inside a right angle. In this case $n = 1$. Consequently, equation (6.1) must be worked out which leads to the well-known result

$$\eta = -\frac{2\sigma q}{g} e^{-\nu y_0}\cos\nu x_0 \cos(\nu x - \sigma t) \tag{7.1}$$

In fact, in the case under consideration we have

$$\varkappa = -1,\quad \alpha = {}^1/_2\pi,\quad \Lambda'(\lambda_0) = 1,\quad c_0' = \rho e^{\mu i},\quad c_1 = -\rho e^{-\mu i}$$

The calculations show that

$$F_j = 1, \qquad A + Bi = \frac{q}{\pi}\{S(c_0') + S(c_1)\}$$

We have

$$A + Bi = \frac{q}{\pi}\int_0^\infty \left(\frac{1}{\zeta - \rho e^{\mu i}} + \frac{1}{\zeta + \rho e^{-\mu i}}\right) e^{i\nu\zeta}\, d\zeta$$

$$A - Bi = \frac{q}{\pi}\int_0^\infty \left(\frac{1}{\zeta - \rho e^{-\mu i}} + \frac{1}{\zeta + \rho e^{\mu i}}\right) e^{-i\nu\zeta}\, d\zeta$$

Hence

$$2Bi = \frac{q}{\pi}\int_{-\infty}^\infty \left(\frac{1}{\zeta - \rho e^{\mu i}} + \frac{1}{\zeta + \rho e^{-\mu i}}\right) e^{i\nu\zeta}\, d\zeta$$

Calculating this integral by means of residues, we obtain

$$B = 2qe^{-\nu y_0}\cos \nu x_0 \qquad (\rho e^{\mu i} = x_0 + iy_0) \tag{7.2}$$

Turning now to formula (6.1), we obtain (7.1).

Let us now take $n = 2$. In this case we have $\alpha = 45°$.

Carrying out the calculations, we find the following expression:

$$-\frac{\sigma}{g\sqrt{2}}(A + B) = -\frac{2\sigma q}{g}\left[e^{-\nu\rho\sin\mu}\cos\left(\frac{1}{4}\pi + \nu\rho\cos\mu\right) + \right.$$
$$\left. + e^{-\nu\rho\cos\mu}\cos\left(\frac{1}{4}\pi + \nu\rho\sin\mu\right)\right] \tag{7.3}$$

for the amplitude of the wave of (6.2).

Let us indicate the values of the various quantities which are necessary to work out formula (5.1) in the case under consideration

$$\varkappa = -i, \qquad a_0 = 1, \qquad a_1 = \frac{1}{\nu}, \qquad \Lambda'(\lambda_0) = \nu\sqrt{2}e^{-1/4\pi i}$$
$$F_0 = 1 - i, \quad F_1 = 2, \quad F_2 = 1 + i, \qquad F_3 = 0$$

Working out formula (5.1), we obtain

$$(A + Bi)\, e^{-1/4\pi i}\pi\sqrt{2}/q = [(1 - i)\, S(c_0') - (1 + i)\, S(c_2)] -$$
$$- i\,[(1 + i)\, S(c_1) + (1 - i)\, S(c_1')]$$

Along with this formula the following formula can also be written, taking the conjugate quantities:

$$(A - Bi)\, e^{1/4\pi i}\pi\sqrt{2}/q = [(1 + i)\, \overline{S}(c_0') - (1 - i)\, \overline{S}(c_2)] +$$
$$+ i\,[(1 - i)\, \overline{S}(c_1) + (1 + i)\, \overline{S}(c_1')]$$

Let us add these two formulas term by term and, using the integral residues, determine the differences between such integrals as $\bar{S}(c_0') - S(c_2)$ which enter into the right-hand side. After carrying this out we obtain equality (7.3).

Let us now assume that $n = 3$; the angle α will be equal to 30°.

Carrying out the calculations, we find the following expression for the amplitude of the waves of (6.3):

$$\sigma A / g = -(2\sigma q / g) \{e^{-\nu\rho \sin (^1/_3\pi - \mu)} \sin [\nu \rho \cos (^1/_3 \pi - \mu)] -$$
$$- \sqrt{3}\, e^{-\nu\rho \sin (^1/_3\pi + \mu)} \cos [\nu\rho \cos (^1/_3\pi + \mu)] + e^{-\nu\rho \sin \mu} \sin (\nu\rho \cos \mu)\} \qquad (7.4)$$

Let us cite the values of the quantities which are needed to work out formula (5.1)

$$\varkappa = {}^1/_2 (1 - i \sqrt{3}), \quad a_0 = 1, \quad a_1 = \nu^{-1} \sqrt{3}, \quad a_2 = \nu^{-2}, \ \Lambda' (\lambda_0) = -i\nu^2 \sqrt{3}$$
$$F_0 = -i \sqrt{3}, \quad F_1 = 3 - i \sqrt{3}, \quad F_2 = 3 + i \sqrt{3}, \quad F_3 = i \sqrt{3}, \quad F_4 = 0, \quad F_5 = 0$$

Let us work out formula (5.1)

$$(A + Bi) \pi / q = i \sqrt{3} [S (c_1') + S (c_2)] - [S (c_2') + S (c_3) - S (c_0') - S (c_1)]$$

Simultaneously with this formula the conjugate formula can be written

$$(A - Bi) \pi / q = -i \sqrt{3} [\bar{S} (c_1') + \bar{S} (c_2)] - [\bar{S} (c_2') + \bar{S} (c_3) - \bar{S} (c_0') - \bar{S} (c_1)]$$

Adding these formulas and evaluating the integrals, we obtain formula (7.4).

Finally, let us take $n = 4$. In this case the angle α will be equal to 22°30′. The calculations reduce to the following expression for the amplitude of the waves of (6.4):

$$\sigma (A - B) / g \sqrt{2} = (8\sigma q / g)\{ (\sqrt{2} + 1)\, e^{-\nu\rho \cos \mu} \sin (^1/_4 \pi - \nu\rho \cos \mu) +$$
$$+ (\sqrt{2} + 1)\, e^{-\nu\rho \sin (^1/_4\pi + \mu)} \sin [^1/_4\pi - \nu\rho \cos (^1/_4 \pi + \mu)] -$$
$$- e^{-\nu\rho \sin \mu} \sin (^1/_4 \pi + \nu\rho \cos \mu) -$$
$$- e^{-\nu\rho \sin (^1/_4\pi - \mu)} \sin [^1/_4 \pi + \nu\rho \cos (^1/_4 \pi - \mu)]\} \qquad (7.5)$$

Let us cite the values of the quantities which are needed to work out formula (5.1); we have

$$\varkappa = {}^1/_2 \sqrt{2}\,(1 - i),\ a_0 = 1,\quad a_1 = \nu^{-1}(\sqrt{2} + 1),\quad a_2 = \nu^{-2}(\sqrt{2} + 1),\ a_3 = \nu^{-3}$$

$$\Lambda'(\lambda_0) = -{}^1/_2 \nu^3 e^{1/_4 \pi i},\quad F_0 = -\sqrt{2}\,(1 + i),\quad F_1 = 2 - 2i\,(\sqrt{2} + 1),\quad F_2 = 4 + 2\sqrt{2}$$

$$F_3 = 2 + 2i\,(\sqrt{2} + 1),\quad F_4 = -\sqrt{2}\,(1 - i),\quad F_5 = 0,\quad F_6 = 0,\quad F_7 = 0$$

Let us represent formula (5.1) and its conjugate in terms of these quantities; adding these formulas term by term, we obtain

$$-\pi / 2q\,(A - B) = (1 + i)\,[\bar{S}(c_4) - S(c_0')] + (1 - i)\,[S(c_4) - \bar{S}(c_0')] +$$
$$+ (1 + i)\,[\bar{S}(c_3') - S(c_1)] + (1 - i)\,[S(c_3') - \bar{S}(c_1)] +$$
$$+ (1 + \sqrt{2})\,(1 - i)\,[\bar{S}(c_3) - \bar{S}(c_1')] + (1 + \sqrt{2})\,(1 + i)\,[S(c_3) - \bar{S}(c_1')] +$$
$$+ (1 + \sqrt{2})\,(1 - i)\,[\bar{S}(c_2') - S(c_2)] + (1 + \sqrt{2})\,(1 + i)\,[S(c_2') - \bar{S}(c_2)]$$

Computing the quantities within the square brackets with the help of residues and carrying out the necessary transpositions, we obtain formula (7.5).

8. Formulas (7.3), (7.4) and (7.5) permit the position of an oscillating source which will not transmit progressive periodic waves to infinity to be found. To determine such source positions it is necessary to solve the equation obtained by setting expressions (7.3), (7.4) and (7.5) equal to zero.

Let us consider the very simple case $\alpha = 45°$; to solve the problem the equation

$$e^{-\nu\rho \sin\mu} \cos({}^1/_4\pi + \nu\rho\cos\mu) + e^{-\nu\rho\cos\mu} \cos({}^1/_4\pi + \nu\rho\sin\mu) = 0$$

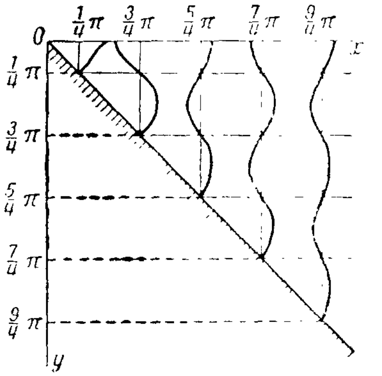
must be investigated.

Let us set $x = \nu\rho\cos\mu$ and $y = \nu\rho\sin\mu$. This equation then reduces to the form

$$e^x \cos({}^1/_4\,\pi + x) = -e^{-y} \cos({}^1/_4\,\pi + y)$$

A curve showing x as a function of y can be drawn from a geometrical construction of the left- and right-hand sides of this equation. This curve consists of an infinite number of separate branches located between the x-axis and the bisector of the coordinate angle, since $x > y$ (Figure).

Let us assume that the angle μ is given; the ratio of x to y is then known to be equal to $\cot \mu > 1$. Let us intersect the constructed curves $x = x(y)$ with the straight line $x = y \cot \mu$; we then obtain an infinite number of points of intersection. The radius vector which connects the origin of the coordinate system with some point of intersection will have a length equal to $\nu\rho$. An infinite set of such quantities $\nu\rho$ having a limiting value of ∞ will be found. Consequently, if along with the angle μ the distance ρ of the source from the origin of the coordinate system is given also, an infinite number of different frequencies tending to infinity will be found which do not generate periodic progressive waves that are transmitted to infinity. If, however μ and ν are given, then an infinite number of different values of ρ are found which together with μ determine an infinite number of source positions with zero amplitude transmitted waves.



It is also possible to regard curves of ρ as a function of μ constructed on a plane within the angle $-\alpha < -\mu < 0$ as the locus of the positions of sources which do not transmit progressive waves to infinity.

BIBLIOGRAPHY

1. Stoker, J.J., *Volny na vode (Water Waves)*. IL, 1959.

2. Wehausen, J.V. and Laitone, E.V., Surface Waves. *Handbuch der Physik*, Vol. 9, 1960.

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